Roll No.							

24-MA-45

M.Sc. IV SEMESTER [MAIN/ATKT] EXAMINATION
JUNE - JULY 2024

MATHEMATICS
Paper - V
[Analytic Number Theory - II]

[Max. Marks : 75] *[Time : 3:00 Hrs.]* *[Min. Marks : 26]*

Note : Candidate should write his/her Roll Number at the prescribed space on the question paper.
Student should not write anything on question paper.
Attempt five questions. Each question carries an internal choice.
Each question carries **15 marks.**

Q. 1 If $F(s) = \Sigma\, f(n)\, n^{-s}$ be absolutely convergent for $\sigma > \sigma_a$ and assume that $f(1) \neq 0$. If $F(\underline{S}) \neq 0$ for $\sigma > \sigma_0 \geq \sigma_a$, then for $\sigma > \sigma_0$, prove that **(15 Marks)**

$F(s) = e^{G(\underline{S})}$

with

$$G(\underline{S}) = \log f(1) + \sum_{n=2}^{\infty} \frac{(f' * f^{-1})(n)}{\log n}\; n^{-s},$$

where f^{-1} is the Dirichlet inverse of f and $f'(n) = f(n) \log(n)$

Also find $G(\underline{S})$ if $\zeta(\underline{S}) = e^{G(\underline{S})}$

OR

Prove that for any Dirichlet series with σ_c finite we have **(15 Marks)**

$0 \leq \sigma_a - \sigma_c \leq 1$

Verify the theorem by an example.

Q. 2 State and prove mean value formula for Dirichlet series and deduce that **(15 Marks)**

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\zeta^{(k)}(\sigma + it)|^2\, dt = \zeta^{(2k)}(2\sigma)$$

OR

State and prove the Perron's formula. **(15 Marks)**

Q. 3 Prove that - **(15 Marks)**

i) $\Gamma(s)\,\Gamma(s + \frac{1}{m}) \ldots \Gamma(s + \frac{m-1}{m}) = (2\pi)^{(m-1)/2}\, m^{(1/2) - ms}\, \Gamma(ms)$,
valid for all s and all integers $m \geq 1$

P.T.O.

ii) Define Hurwitz zeta function and show that the Riemann zeta function and Dirichlet L-function can be expressed in terms of Hurwitz zeta function.

OR

Prove that the series for $\zeta(s, a)$ converges absolutely for $\sigma > 1$. The convergence is uniform in every half - plane $\sigma \geq 1 + \delta$, $\delta > 0$, so $\zeta(s, a)$ is analytic function of S in the half - plane $\sigma > 1$. **(15 Marks)**

Q. 4 Obtain the functional equation for Dirichlet L - functions. **(15 Marks)**

OR

Prove that - **(15 Marks)**

i) The Riemann zeta function $\zeta(\underline{S})$ is analytic every where except for a simple pole at s = 1 with residue 1.

ii) For the principal character χ_1 mod k, the function $L(s, \chi_1)$ is analytic every where except for a simple pole at s = 1 with residue $\phi(k) / k$.

iii) If $\chi \neq \chi_1$, $L(s, \chi)$ is an entire function of s.

Q. 5 Prove that if $n \geq 2$ we have **(15 Marks)**

$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k$$

compute the Bernoulli numbers for n = 1, 2, ... , 10.

OR

If k is a positive integer , then prove that **(15 Marks)**

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k} B_{2k}}{2(2k)!}$$

and hence show that B_{2k} is alternate in sign.

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